Tangle diagrams

Let T be a tangle diagrams
- $P_T :=$ vector space generated by endpoints of T
- $V_T :=$ vector space generated by exterior faces of T
- $U_T :=$ vector space generated by interior faces of T
- $M_T :=$ quadratic form on $U_T \oplus V_T$ by [doing a thing at each crossing]

Let $\delta : V_T \hookrightarrow P_T$ be the map that embeds V_T into P_T by sending each face to its first endpoint (going counterclockwise).

Operations on tangles:
- **Disjoint union**: $T_1, T_2 \mapsto T_1 \sqcup T_2 = T_1 \sqcup_{p_1, p_2} T_2$
- **Capping**: $T \mapsto \widetilde{T} = \widetilde{T}_p$

3-tuples

Let P be a vector space. Let $\mathcal{T}(P)$ be 3-tuples $\{(n, W, B)\}$ where:
- $n \in \mathbb{Z}$
- $W \subseteq P$ a subspace
- $B : W \to W^*$ a quadratic form

Operations on 3-tuples:
- Adding: $(n_1, W_1, B_1) + (n_2, W_2, B_2) := (n_1 + n_2, W_1 \oplus W_2, B_1 \oplus B_2) \in \mathcal{T}(P_1 \oplus P_2)$.
- Pullback by $\phi : P_1 \to P_2$: $\phi^*(n_2, W_2, B_2) = (n_2, \phi^{-1}(W_2), \phi^* B_2 \phi)$

A map s from tangle diagrams to 3-tuples

Let s be a map that sends T to $s(T) = (n_T, W_T, B_T) \in \mathcal{T}(P_T)$ where:
- $n_T = \sigma(M|_{U_T})$
- $W_T = \{v \in V_T : M(v)|_{U_T} \in \text{im}(M|_{U_T})\}$
- $B_T : W_T \to W_T^*$ is given by $B_T(v_1)(v_2) := M(v_1 - u_1)(v_2 - u_2)$ where $u_1, u_2 \in U_T$ are such that $M(u_1)|_{U_T} = M(v_1)|_{U_T}$ and $M(u_2)|_{U_T} = M(v_2)|_{U_T}$

Note that:
- B_T does not depend on the choices of u_1, u_2.
- Can decompose $U^* \cong \text{im}(M|_U) \oplus \ker(M|_U)^*$ so can also describe W_T as:
    - $\{v \in V_T : M(v)|_{\ker(M|_U)} = 0\}$ or
    - the image of the projection of $\{w \in U_T \oplus V_T : M(w)|_U = 0\}$ to V_T.

How operations on tangles change s:
- For disjoint union: $s(T_1 \sqcup T_2) = s(T_1) \oplus s(T_2)$
- For capping:
    - If there is $v \in W_T$ that becomes an interior face in $\widetilde{T}$, then $s(\widetilde{T}) = i^*(n_T + B_T(v)(v), \text{Ann}_{B_T}(v), B_T|_{\text{Ann}_{B_T}(v)})$
    - If there is no such v, then $s(\widetilde{T}_p) = i^*(s(T))$
  where $i : P_{\widetilde{T}} \to P_T$ is the identity everywhere except $p_a \mapsto p_a + p_b$ where p_a, p_b are the endpoints before and after p.

Theorem 0.1. *If $s(T_1) = s(T_2)$, then:*
- $s(T_1 \sqcup T) = s(T_2 \sqcup T)$ *for any T*
- $s(\widetilde{T_1}) = s(\widetilde{T_2})$

A tangle invariant

Theorem 0.2. *s is a tangle invariant.*

Proof. By the previous theorem, just need to show that applying s to the two sides of each Reidemeister move gives the same thing. $\square$

§4

$$\frac{1}{2}x^T A x + x^T B y = \frac{1}{2}(x+A^{-1}By)^T A(x+A^{-1}By) - \frac{1}{2}y^T B^T A^{-1} B y$$

$$x^T B y =$$

$$(P^T B)^T = B^T P$$

$$xy = \frac{(x+y)^2}{2} - \frac{(x-y)^2}{2}$$

$$y^T B^T x$$

$$\mathrm{sign}\begin{pmatrix} O & B \\ B^T & O \end{pmatrix} \overset{?}{=} O$$

1. row & col reduce
   B to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$(x+Py)^T B y = x^T B y + y^T P^T B y$$

2. compute $\mathrm{sign}\left(\sum x_i y_i\right)$.

What best can be made of the inverse of a quadratic form, if it is degenerate?

Signature: Additive
inv. under bilinear
pullbacks.

$$\mathrm{sign}\begin{pmatrix} O & 1 \\ 1 & O \end{pmatrix} = O$$

$$\begin{matrix} O & 1 & -1 \\ 1 & b & O \\ -1 & O & -b \end{matrix} \rightarrow$$